DISCRETE COCOMPACT SUBGROUPS OF $\mathrm{G}_{5,3}$ AND RELATED C^*-ALGEBRAS

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Abstract. The discrete cocompact subgroups of the 5-dimensional Lie group $\mathrm{G}_{5,3}$ are determined up to isomorphism. Each of their group C^*-algebras is studied by determining all of its simple infinite dimensional quotient C^*-algebras. The K-groups and trace invariants of the latter are also obtained.

§1. Introduction.

Consider the Lie group $\mathrm{G}_{5,3}$ equal to $\mathbb{R}^5$ as a set with multiplication given by

$$(h,j,k,m,n)(h',j',k',m',n') =$$
$$(h+h'+nj'+m'n(n-1)/2+mk', j+j'+nm', k+k', m+m', n+n').$$

and inverse

$$(h,j,k,m,n)^{-1} = (-h+nj+mk-mn(n-1)/2, -j+nm, -k, -m, -n).$$

The group $\mathrm{G}_{5,3}$ is one of only six nilpotent, connected, simply connected, 5-dimensional Lie groups; it seemed the most tractable of them for our present purposes. (Our notation is as in Nielsen [7], where a detailed catalogue of Lie groups like this one is given.) In [5, Section 3] the authors have studied a natural discrete cocompact subgroup $\mathrm{H}_{5,3}$, the lattice subgroup $\mathrm{H}_{5,3} = \mathbb{Z}^5 \subset \mathrm{G}_{5,3}$. In section 2 of this paper we study the group $\mathrm{G}_{5,3}$ more closely, determining the isomorphism classes of all its discrete cocompact subgroups (Theorem 1). These are given by five integer parameters $\alpha, \beta, \gamma, \delta, \epsilon$ that satisfy certain conditions (see $(*)$ and $(**)$ of Theorem 1), and are denoted by $\mathrm{H}_{5,3}(\alpha,\beta,\gamma,\delta,\epsilon)$. It is shown that each such subgroup is isomorphic to a cofinite subgroup of $\mathrm{H}_{5,3} = \mathrm{H}_{5,3}(1,0,1,1,0)$. Conversely, each cofinite subgroup of $\mathrm{H}_{5,3} \subset \mathrm{G}_{5,3}$ is a discrete cocompact subgroup of $\mathrm{G}_{5,3}$. In sections 3 and 4 the group C^*-algebras of the $\mathrm{H}_{5,3}(\alpha,\beta,\gamma,\delta,\epsilon)$'s are examined by obtaining their simple infinite dimensional quotients. These are shown to be crossed products of certain types of Heisenberg C^*-algebras (in Packer's terminology [10]) and the rest are matrix algebras over irrational rotation algebras (Theorem 5). In section 5 the K-groups of the simple quotients are calculated (Theorem 6) as are their trace invariants (Theorem 8). The paper ends with a discussion of the classification of the simple quotients.

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We use the conventional notation for crossed products as in, for example, [11] or [16]. Hence, if a discrete group G acts on a C^*-algebra A, we write $C^*(A, G)$ to denote the associated C^*-crossed product algebra. We use a similar notation for twisted crossed products, i.e. when there is a cocycle instead of an action (as in Theorem 2). (See the Preliminaries of [5] for more details.)

§2. Determination of the Discrete Cocompact Subgroups.

1. Theorem. *Every discrete cocompact subgroup* H *of* $\mathrm{G}_{5,3}$ *has the following form: there are integers* α, β, γ, δ *and* ϵ *satisfying* $\alpha, \gamma, \delta > 0$, *and*

$$(*) \qquad 0 \le \epsilon \le \gcd\{\gamma, \delta\}/2 \text{ and}$$

$$(**) \qquad 0 \le \beta \le \gcd\{\alpha, \gamma, \delta, \epsilon\}/2,$$

yielding $\mathrm{H} \cong \mathrm{H}_{5,3}(\alpha, \beta, \gamma, \delta, \epsilon)$ $(= \mathbb{Z}^5$ *as a set) with multiplication*

$$(\mathrm{m}) \qquad \begin{cases} (h, j, k, m, n)(h', j', k', m', n') = \\ (h + h' + \gamma n j' + \alpha\gamma m' n(n-1)/2 + \beta n m' + \delta m k' + \epsilon n k', \\ j + j' + \alpha n m', k + k', m + m', n + n'). \end{cases}$$

Different choices for α, β, γ, δ *and* ϵ *give non-isomorphic groups. Each such group is, in fact, isomorphic to a cofinite subgroup of* $\mathrm{H}_{5,3}$ *(the lattice subgroup of* $\mathrm{G}_{5,3}$*), and each cofinite subgroup of* $\mathrm{H}_{5,3}$ *is isomorphic to some* $\mathrm{H}_{5,3}(\alpha, \beta, \gamma, \delta, \epsilon)$.

PROOF. Using the discreteness and cocompactness as in [6], the second commutator subgroup of H tells us that there is a member (with entries that don't need to be identified indicated by $*$)

$e_5 = (*, *, *, \mathfrak{a}, z)$

of H, where $z > 0$ is the smallest positive number that can appear as the last coordinate of a member of H. Continuing in this vein, we get

$e_4 = (*, *, *, y, 0)$,

$e_3 = (*, \mathfrak{b}, x, 0, 0)$,

$e_2 = (*, w, 0, 0, 0)$ and

$e_1 = (v, 0, 0, 0, 0)$,

where $x > 0$ is the smallest positive number that can appear as the 3rd coordinate of a member of H whose last 2 coordinates are 0, and similarly for v, w and y. Also, all other coordinates are ≥ 0, and the bottom non-zero coordinate in each column is greater than the coordinates above it, e.g., $w > \mathfrak{b} \ge 0$ and w is also greater than the 2nd coordinate of e_5 or of e_4. These considerations show that the map

$$\pi : (h, j, k, m, n) \mapsto e_1^h e_2^j e_3^k e_4^m e_5^n, \ \ \mathbb{Z}^5 \to \mathrm{H},$$

is $1-1$ and onto. We want the multiplication (m) for $\mathbb{Z}^5$ that makes π a homomorphism (hence an isomorphism); (m) is determined using the commutators,

$$(C) \qquad \begin{cases} [e_5, e_4] = (*, zy, 0, 0, 0) = e_1^{\beta} e_2^{\alpha}, \ \ [e_5, e_3] = (z\mathfrak{b} + x\mathfrak{a}, 0, 0, 0, 0) = e_1^{\epsilon}, \\ [e_5, e_2] = (zw, 0, 0, 0) = e_1^{\gamma}, \text{ and } [e_4, e_3] = (xy, 0, 0, 0) = e_1^{\delta}, \end{cases}$$

for some integers $\alpha, \beta, \gamma, \delta, \epsilon$ (other pairs of e's commuting). Using the commutators to collect terms in

$$(e_1^h e_2^j e_3^k e_4^m e_5^n)(e_1^{h'} e_2^{j'} e_3^{k'} e_4^{m'} e_5^{n'})$$

gives the multiplication formula (m) for $\mathbb{Z}^5$, and also the equation

$$e_5^n e_4^{m'} = e_1^{\alpha\gamma m' n(n-1)/2 + \beta n m'} e_2^{\alpha m' n} e_4^{m'} e_5^n,$$

which the reader may find helpful in checking computations later.

For a start in putting the restrictions on $\alpha, \beta, \gamma, \delta, \epsilon$, (C) tells us that $\alpha, \gamma, \delta > 0$ (since v, w, x, y and $z > 0$). Let Z denote the center of $\mathrm{H}_{5,3}(\alpha, \beta, \gamma, \delta, \epsilon)$, $Z = (\mathbb{Z}, 0, 0, 0, 0)$. Then, as for G_4, with quotients and subgroups it is shown that different (positive) α, γ, δ give non-isomorphic groups, e.g., $\mathrm{H}_{5,3}(\alpha, \beta, \gamma, \delta, \epsilon)/Z$ gives α, then Z modulo the second commutator subgroup gives γ, and also, with K_3, $K_4 \subset \mathrm{H}_{5,3}(\alpha, \beta, \gamma, \delta, \epsilon)$ as defined below,

$$Z \supset (\delta\mathbb{Z}, 0, 0, 0, 0) = \{xyx^{-1}y^{-1} \mid x \in K_3, \ y \in K_4\}$$

and $Z/(\delta\mathbb{Z}, 0, 0, 0, 0) = \mathbb{Z}_\delta$, the cyclic group of order δ.

Then we have an isomorphism of $\mathrm{H}_{5,3}(\alpha, \beta, \gamma, \delta, \epsilon)$ onto $\mathrm{H}_{5,3}(\alpha, \beta, \gamma, \delta, \epsilon + d\gamma + e\delta)$, which is simpler to give in terms of generators,

$$(\circledast) \qquad e_3 \mapsto e_3' = e_2^d e_3, \ \ e_5 \mapsto e_5' = e_4^e e_5, \ \text{and } e_i \mapsto e_i' = e_i \text{ otherwise.}$$

Here we are merely changing the basis for $\mathrm{H}_{5,3}(\alpha, \beta, \gamma, \delta, \epsilon)$, and the only commutator (using (m) and (C)) that changes is $[e_5', e_3'] = e_1^{\epsilon + e\delta + d\gamma}$, so the resulting isomorphism is of $\mathrm{H}_{5,3}(\alpha, \beta, \gamma, \delta, \epsilon)$ onto $\mathrm{H}_{5,3}(\alpha, \beta, \gamma, \delta, \epsilon + d\gamma + e\delta)$, which shows we can require

$$0 \le \epsilon < \gcd\{\gamma, \delta\}.$$

This, accompanied by another isomorphism,

$$(\circledast') \qquad (h, j, k, m, n) \mapsto (-h, -j, k, -m, n), \ \ \mathrm{H}_{5,3}(\alpha, \beta, \gamma, \delta, \epsilon) \to \mathrm{H}_{5,3}(\alpha, \beta, \gamma, \delta, -\epsilon),$$

assures that we can have

$$(*) \qquad 0 \le \epsilon \le \gcd\{\gamma, \delta\}/2,$$

the required range for ϵ.

Now, to control β,

$$(\dagger) \qquad \begin{cases} e_1 \mapsto e_1 = e_1', \ \ e_2 \mapsto e_1^{-q} e_2 = e_2', \ \ e_3 \to e_3 = e_3', \\ e_4 \mapsto e_2^r e_3^g e_4 \text{ and } e_5 \mapsto e_3^{-f} e_5 = e_5' \end{cases}$$

is an isomorphism of $\mathrm{H}_{5,3}(\alpha, \beta, \gamma, \delta, \epsilon)$ onto $\mathrm{H}_{5,3}(\alpha, \beta + q\alpha + r\gamma + f\delta + g\epsilon, \gamma, \delta, \epsilon)$, which yields

$$0 \le \beta < \gcd\{\alpha, \gamma, \delta, \epsilon\}.$$

Then the isomorphism

$$(\dagger') \qquad (h,j,k,m,n) \mapsto (-h,j,k,-m,-n)$$

of $\mathrm{H}_{5,3}(\alpha,\beta,\gamma,\delta,\epsilon)$ onto $\mathrm{H}_{5,3}(\alpha,-\beta+\alpha\gamma,\gamma,\delta,\epsilon)$ leads to the conclusion

$$(**) \qquad 0 \le \beta \le \gcd\{\alpha,\gamma,\delta,\epsilon\}/2.$$

It must still be shown that changing ϵ or β within the allowed limits (namely, ϵ and β must satisfy $(*)$ and $(**)$, respectively) gives a non-isomorphic group.

So, suppose that $\varphi : \mathrm{H} = \mathrm{H}_{5,3}(\alpha,\beta,\gamma,\delta,\epsilon) \to \mathrm{H}_{5,3}(\alpha,\beta',\gamma,\delta,\epsilon') = \mathrm{H}'$ is an isomorphism. Then

$$\varphi : Z = K_1 = (\mathbb{Z},0,0,0,0) \to (\mathbb{Z},0,0,0,0) = K_1' = Z',$$

$$K_2 = (\mathbb{Z},\mathbb{Z},0,0,0) \to (\mathbb{Z},\mathbb{Z},0,0,0) = K_2',$$

$$K_3 = (\mathbb{Z},\mathbb{Z},\mathbb{Z},0,0) \to (\mathbb{Z},\mathbb{Z},\mathbb{Z},0,0) = K_3', \text{ and}$$

$$K_4 = (\mathbb{Z},\mathbb{Z},\mathbb{Z},\mathbb{Z},0) \to (\mathbb{Z},\mathbb{Z},\mathbb{Z},\mathbb{Z},0) = K_4',$$

since the Z's are the centers, the K_2's consist of those $s \in \mathrm{H}$ for which s^r is in the commutator subgroup of H for some $r \in \mathbb{Z}$, the K_3's are the largest subsets for which all commutators are central (e.g., $xyx^{-1}y^{-1} \in Z$ for all $x \in K_3$ and $y \in \mathrm{H}_{5,3}(\alpha,\beta,\gamma,\delta,\epsilon)$), and the K_4's are the centralizers of the commutator subgroups. So we must have

$$\varphi(0,0,0,0,1) = (*,*,-f,e,a) = S_5 \text{ with } a = \pm 1,$$

$$\varphi(0,0,0,1,0) = (*,r,g,b,0) = S_4 \text{ with } b = \pm 1, \text{ and}$$

$$\varphi(0,0,1,0,0) = (*,d,c,0,0) = S_3 \text{ with } c = \pm 1;$$

furthermore, commutators give

$$\varphi(\beta,\alpha,0,0,0) = [S_5,S_4] = S_5S_4S_5^{-1}S_4^{-1} = (*,\alpha ab,0,0,0),$$

hence $\varphi(0,1,0,0,0) = (q,ab,0,0,0) = S_2$, and

$$\varphi(\gamma,0,0,0,0) = [S_5,S_2] = (\gamma a^2 b,0,0,0,0),$$

so $\varphi(1,0,0,0,0) = (b,0,0,0,0) = S_1$, but also

$$\varphi(\delta,0,0,0,0) = [S_4,S_3] = (\delta bc,0,0,0,0),$$

so $c = 1$. Furthermore, $\varphi(\epsilon,0,0,0,0) = [S_5,S_3] = (a\epsilon' + e\delta + ad\gamma,0,0,0,0)$, which shows that the manipulations at $(\circledast)$ and $(\circledast')$ above give the only way of changing ϵ in $\mathrm{H}_{5,3}(\alpha,\beta,\gamma,\delta,\epsilon)$; that is, if

$$(*) \qquad 0 \le \epsilon, \epsilon' \le \gcd\{\gamma,\delta\}/2$$

and $\epsilon = \pm\epsilon' + a_1\delta + a_2\gamma$ with $a_1, a_2 \in \mathbb{Z}$, then $\epsilon = \epsilon'$. Now consider

$$\varphi(h,j,k,m,n) = \varphi((h,0,0,0,0)(0,j,0,0,0)(0,0,k,0,0)(0,0,0,m,0)(0,0,0,0,n))$$

$$= (hS_1)\cdot(jS_2)\cdot(kS_3)\cdot S_4^m \cdot S_5^n = hS_1 + jS_2 + kS_3 + S_4^m\cdot S_5^n \in \mathrm{H}'.$$

Note that $S_5^n \neq nS_5$, but $S_5^n = (*,*,-nf,ne,na)$, and also $S_4^m = (*,mr,mg,mb,0)$; further, the (jS_2) term puts a jq in the first entry of $\varphi(h,j,k,m,n)$, so also $(j+j'+\alpha nm')q$ in the first entry of $\varphi(h,j,k,m,n)\cdot\varphi(h',j',k',m',n')$ (product in $\mathrm{H}_{5,3}(\alpha,\beta',\gamma,\delta,\epsilon)$). Then, equating the coefficients of the nm' terms in the first entry of

$$\varphi(e_5^n e_4^{m'}) \text{ and } \varphi(e_5^n)\varphi(e_4^{m'}) = S_5^n S_4^{m'} \text{ gives}$$

$$b(-\alpha\gamma/2+\beta) + q\alpha = ab\beta' - ab\alpha\gamma/2 + ag\epsilon + ar\gamma + (eg+bf)\delta, \text{ or}$$

$$\beta = \pm\beta' + a_1\alpha + a_2\gamma + a_3\delta + a_4\epsilon \text{ for some } a_i \in \mathbb{Z},\ \ 1 \le i \le 4,$$

which shows that the manipulations at (†) and (†′) above give the only way of changing just β in $\mathrm{H}_{5,3}(\alpha,\beta,\gamma,\delta,\epsilon)$.

Here is an isomorphism φ of $\mathrm{H}_{5,3}(\alpha,\beta,\gamma,\delta,\epsilon)$ into the lattice subgroup $\mathrm{H}_{5,3} = \mathbb{Z}^5 \subset \mathrm{G}_{5,3}$ in terms of generators; $\mathrm{H}_{5,3}$ has multiplication

$$(\mathrm{m}') \qquad \begin{cases} (h,j,k,m,n)(h',j',k',m',n') = \\ (h+h'+nj'+m'n(n-1)/2+mk', \\ j+j'+nm', k+k', m+m', n+n') \end{cases}$$

(i.e., $\alpha = \gamma = \delta = 1$ and $\beta = \epsilon = 0$). First suppose $\epsilon > 0$. Then, with $\mathfrak{d} = \alpha\gamma\epsilon$ and generators

$$e_1 = (1,0,0,0,0),\ \ e_2 = (0,1,0,0,0),\ \ \ldots\ ,\ \ e_5 = (0,0,0,0,1)$$

for $\mathrm{H}_{5,3}(\alpha,\beta,\gamma,\delta,\epsilon)$ satisfying

$$(C) \qquad [e_5,e_4] = e_1^\beta e_2^\alpha,\ \ [e_5,e_3] = e_1^\epsilon,\ \ [e_5,e_2] = e_1^\gamma, \text{ and } [e_4,e_3] = e_1^\delta,$$

φ is given by

$$\varphi : e_1 \mapsto e_1' = (\delta\mathfrak{d}^2,0,0,0,0),\ \ e_2 \mapsto e_2' = (\gamma\delta\mathfrak{d}(\mathfrak{d}-1)/2, \gamma\delta\mathfrak{d},0,0,0),$$

$$e_3 \mapsto e_3' = (0,\delta\epsilon\mathfrak{d},\delta\epsilon\mathfrak{d},0,0),\ \ e_4 \mapsto e_4' = (0,\beta\delta\mathfrak{d},0,\alpha\gamma\delta,0),$$

$$\text{and } e_5 \mapsto e_5' = (0,0,0,0,\mathfrak{d}).$$

That φ is an isomorphism is verified by showing that $\{e_1',e_2',e_3',e_4',e_5'\} \subset \mathrm{H}_{5,3}$ satisfy (C). (Here φ is given by

$$(h,j,k,m,n) \mapsto (\delta\mathfrak{d}^2 h + (\gamma\delta\mathfrak{d}(\mathfrak{d}-1)/2)j, \gamma\delta\mathfrak{d}j + \delta\epsilon\mathfrak{d}k + \beta\delta\mathfrak{d}m, \delta\epsilon\mathfrak{d}k, \alpha\gamma\delta m, \mathfrak{d}n).)$$

When $\epsilon = 0$, use $\mathfrak{d} = \alpha\gamma$ and $e_3' = (0,0,\delta\mathfrak{d},0,0)$.

It is easy to see that the image $\mathrm{H}_1 = \varphi(\mathrm{H}_{5,3}(\alpha,\beta,\gamma,\delta,\epsilon))$ is cofinite in $\mathrm{H}_{5,3}$. Consider the coset $s\mathrm{H}_1$ for $s = (h,j,k,m,n) \in \mathrm{H}_{5,3}$; since $e_5' = (0,0,0,0,\mathfrak{d})$, we can choose $r_5 \in \mathbb{Z}$ so that $s{e_5'}^{r_5}$ has its last coordinate in $[0,\mathfrak{d})$. Then choose $r_4 \in \mathbb{Z}$ so that $s{e_5'}^{r_5}{e_4'}^{r_4}$ has its second last coordinate in $[0,\alpha\gamma\delta)$. Continuing like this, we arrive at

$$s{e_5'}^{r_5}{e_4'}^{r_4}{e_3'}^{r_3}{e_2'}^{r_2}{e_1'}^{r_1} \in$$

$$K = \big([0,\delta\mathfrak{d}^2) \times [0,\gamma\delta\mathfrak{d}) \times [0,\delta\epsilon\mathfrak{d}) \times [0,\alpha\gamma\delta) \times [0,\mathfrak{d})\big) \cap \mathbb{Z}^5 \subset \mathrm{H}_{5,3},$$

so every coset $s\mathrm{H}_1$, $s \in \mathrm{H}_{5,3}$, has a representative in K, which is a finite set. It follows that the quotient map $\mathrm{H}_{5,3} \to \mathrm{H}_{5,3}/\mathrm{H}_1$ maps K onto $\mathrm{H}_{5,3}/\mathrm{H}_1$, which is therefore finite. (A similar argument shows that $\mathrm{G}_{5,3}/\mathrm{H}_1$ is cocompact.)

Finally, note that since any cofinite subgroup of $\mathrm{H}_{5,3}$ is also a discrete cocompact subgroup of $\mathrm{G}_{5,3}$, it must therefore be isomorphic to some $\mathrm{H}_{5,3}(\alpha,\beta,\gamma,\delta,\epsilon)$. This completes the proof. $\square$

REMARKS. 1. The image $\mathrm{H}_1 = \varphi(\mathrm{H}_{5,3}(\alpha,\beta,\gamma,\delta,\epsilon))$ above is not a normal subgroup of $\mathrm{H}_{5,3}$, e.g.,

$$(0,0,1,0,0)e_5'(0,0,-1,0,0) = (\mathfrak{d},0,0,0,0) \notin \mathrm{H}_1.$$

This makes it seem unlikely that $\mathrm{H}_{5,3}(\alpha,\beta,\gamma,\delta,\epsilon)$ can be embedded in $\mathrm{H}_{5,3}$ as a normal subgroup; however, the existence of such an embedding is still a possiblity.

2. The theorem gives an isomorphism φ of $\mathrm{H}_{5,3}(\alpha,\beta,\gamma,\delta,\epsilon)$ into $\mathrm{H}_{5,3}$; conversely, there is always an isomorphism φ' of $\mathrm{H}_{5,3}$ into $\mathrm{H}_{5,3}(\alpha,\beta,\gamma,\delta,\epsilon)$, and as for φ, it is easier to give φ' in terms of the generators e_i, $1 \le i \le 5$, of $\mathrm{H}_{5,3}$, which satisfy

$$(C') \qquad [e_5,e_4] = e_2, \ \ [e_5,e_2] = e_1 = [e_4,e_3].$$

Then

$$\varphi' : e_1 \mapsto e_1' = (\alpha\gamma^2\delta^2,0,0,0,0), \ \ e_2 \mapsto e_2' = (\alpha\gamma^2\delta(\delta-1)/2, \alpha\gamma\delta,0,0,0),$$

$$e_3 \mapsto e_3' = (0,-\alpha\delta\epsilon,\alpha\delta\gamma,0,0), \ \ e_4 \mapsto e_4' = (0,-\beta,0,\gamma,0),$$

$$\text{and } e_5 \mapsto e_5' = (0,0,0,0,\delta).$$

That φ' is an isomorphism is verified by showing that $\{e_1',e_2',e_3',e_4',e_5'\} \subset \mathrm{H}_{5,3}(\alpha,\beta,\gamma,\delta,\epsilon)$ satisfy (C'). (Here φ' is given by

$$(h,j,k,m,n) \mapsto (\alpha\gamma^2\delta^2 h + j\alpha\gamma^2\delta(\delta-1)/2, \alpha\gamma\delta j - \alpha\delta\epsilon k - \beta m, \alpha\gamma\delta k, \gamma m, \delta n).)$$

So, as for the 3-dimensional groups $\mathrm{H}_3(p)$ and the 4-dimensional groups $\mathrm{H}_4(p_1,p_2,p_3)$, here we have an infinite family of non-isomorphic groups, each of which is isomorphic to a subgroup of any other one.

§3. Infinite Dimensional Simple Quotients of $C^*(\mathrm{H}_{5,3}(\alpha,\beta,\gamma,\delta,\epsilon))$.

We begin by obtaining concrete representations on $L^2(\mathbb{T}^2)$ of the faithful simple quotients (i.e., those arising from a faithful representation of $\mathrm{H}_{5,3}(\alpha,\beta,\gamma,\delta,\epsilon)$), and consider first the case $\epsilon=0$. In this case $\mathrm{H}_{5,3}(\alpha,\beta,\gamma,\delta,0)$ has an abelian normal subgroup $N=(\mathbb{Z},\mathbb{Z},0,\mathbb{Z},0)$, with quotient

$$\mathrm{H}_{5,3}(\alpha,\beta,\gamma,\delta,0)/N\cong(0,0,\mathbb{Z},0,\mathbb{Z})=\mathbb{Z}^2,$$

also abelian and embedded in $\mathrm{H}_{5,3}(\alpha,\beta,\gamma,\delta,0)$ as a subgroup, so that $\mathrm{H}_{5,3}(\alpha,\beta,\gamma,\delta,0)$ is isomorphic to a semidirect product $N\times\mathbb{Z}^2$; in this situation, the simple quotients of $C^*(\mathrm{H}_{5,3}(\alpha,\beta,\gamma,\delta,0))$ can be presented as C^*-crossed products using flows from commuting homeomorphisms, as follows.

Let $\lambda=e^{2\pi i\theta}$ for an irrational θ, and consider the flow $\mathcal{F}'=(\mathbb{Z}^2,\mathbb{T}^2)$ generated by the commuting homeomorphisms

$$\psi_1':(w,v)\mapsto(\lambda^\gamma w,\lambda^\beta w^\alpha v)\quad\text{and}\quad\psi_2':(w,v)\mapsto(w,\lambda^{-\delta}v).$$

$\mathcal{F}'$ is minimal, so the C^*-crossed product $\mathcal{C}'=C^*(\mathcal{C}(\mathbb{T}^2),\mathbb{Z}^2)$ is simple [1, Corollary 5.16].

Let v and w denote (as well as members of $\mathbb{T}$) the functions in $\mathcal{C}(\mathbb{T}^2)$ defined by

$$(w,v)\mapsto v\text{ and }w,$$

respectively. Define unitaries U, V, W and X on $L^2(\mathbb{T}^2)$ by

$$(\mathcal{U}')\qquad U:f\mapsto f\circ\psi_1',\quad V:f\mapsto vf,\quad W:f\mapsto f\circ\psi_2'\ \text{ and }\ X:f\mapsto wf.$$

These unitaries satisfy

$$(\mathrm{CR}')\qquad UV=\lambda^\beta X^\alpha VU,\quad UX=\lambda^\gamma XU,\quad\text{and}\quad VW=\lambda^\delta WV$$

(other pairs of unitaries commuting), equations which ensure that

$$\pi:(h,j,k,m,n)\mapsto\lambda^h X^j W^k V^m U^n$$

is a representation of $\mathrm{H}_{5,3}(\alpha,\beta,\gamma,\delta,0)$. Denote by $\mathrm{A}_\theta^{5,3}(\alpha,\beta,\gamma,\delta,0)$ the C^*-subalgebra of $B(L^2(\mathbb{T}^2))$ generated by π, i.e., by U, V, W and X. Since $\mathrm{A}_\theta^{5,3}(\alpha,\beta,\gamma,\delta,0)$ is generated by a representation of $\mathrm{H}_{5,3}(\alpha,\beta,\gamma,\delta,0)$, it is a quotient of the group C^*-algebra $C^*(\mathrm{H}_{5,3}(\alpha,\beta,\gamma,\delta,0))$. It follows readily that $\mathrm{A}_\theta^{5,3}(\alpha,\beta,\gamma,\delta,0)$ is isomorphic to the simple C^*-crossed product $\mathcal{C}'$ above, and hence is simple.

However, when $0<\epsilon\le\gcd\{\gamma,\delta\}/2$ (which implies $\gamma>1$, by $(*)$), $\mathrm{H}_{5,3}(\alpha,\beta,\gamma,\delta,\epsilon)$ is only an extension $(\mathbb{Z},\mathbb{Z},0,\mathbb{Z},0)\times(0,0,\mathbb{Z},0,\mathbb{Z})=N\times\mathbb{Z}^2$, and not a semidirect product. Nonetheless, we can modify the flow $\mathcal{F}'$ representing $\mathrm{A}_\theta^{5,3}(\alpha,\beta,\gamma,\delta,0)$ above to get a concrete representation of $\mathrm{A}_\theta^{5,3}(\alpha,\beta,\gamma,\delta,\epsilon)$. Consider the flow $\mathcal{F}=(\mathbb{Z}^2,\mathbb{T}^2)$ generated by the commuting homeomorphisms

$$\psi_1:(w,v)\mapsto(\lambda w,\lambda^\beta w^{\alpha\gamma}v)\text{ and }\psi_2:(w,v)\mapsto(w,\lambda^{-\delta}v).$$

$\mathcal{F}$ is minimal, so the C^*-crossed product $\mathcal{C} = C^*(\mathcal{C}(\mathbb{T}^2), \mathbb{Z}^2)$ is simple. Define unitaries on $L^2(\mathbb{T}^2)$ by

$$(\mathcal{U}) \qquad U : f \mapsto f \circ \psi_1, \quad V : f \mapsto vf, \quad W : f \mapsto w^\epsilon f \circ \psi_2 \quad \text{and} \quad X : f \mapsto w^\gamma f.$$

These unitaries satisfy

$$(\mathrm{CR}) \qquad UV = \lambda^\beta X^\alpha VU, \quad UX = \lambda^\gamma XU, \quad VW = \lambda^\delta WV \quad \text{and} \quad UW = \lambda^\epsilon WU,$$

equations which ensure that

$$\pi : (h, j, k, m, n) \mapsto \lambda^h X^j W^k V^m U^n$$

is a representation of $\mathrm{H}_{5,3}(\alpha, \beta, \gamma, \delta, \epsilon)$. Denote by $\mathrm{A}_\theta^{5,3}(\alpha, \beta, \gamma, \delta, \epsilon)$ the C^*-subalgebra of $B(L^2(\mathbb{T}^2))$ generated by π. Now $\mathrm{A}_\theta^{5,3}(\alpha, \beta, \gamma, \delta, \epsilon)$ is isomorphic only to a subalgebra of $\mathcal{C}$ (as may be shown using conditional expectations); a unitary that is missing is $X' : f \mapsto wf$ (since $\gamma > 1$).

NOTE. The reason we did not use $\mathcal{F}$ when $\epsilon = 0$ (and $\gamma > 1$) is that $\mathrm{A}_\theta^{5,3}(\alpha, \beta, \gamma, \delta, 0)$ seems to be isomorphic only to a subalgebra of $\mathcal{C}$ in that case too, whereas with $\mathcal{F}'$, $\mathrm{A}_\theta^{5,3}(\alpha, \beta, \gamma, \delta, 0) \cong \mathcal{C}'$.

Since the flow method can no longer be used to prove the simplicity of the algebra $\mathrm{A}_\theta^{5,3}(\alpha, \beta, \gamma, \delta, \epsilon)$ (when $0 < \epsilon \le \gcd\{\gamma, \delta\}/2$), we use the strong result of Packer [9].

2. Theorem. *Let $\lambda = e^{2\pi i\theta}$ for an irrational θ.*
(a) There is a unique (up to isomorphism) simple C^*-algebra $\mathrm{A}_\theta^{5,3}(\alpha, \beta, \gamma, \delta, \epsilon)$ generated by unitaries U, V, W and X satisfying

$$(\mathrm{CR}) \qquad UV = \lambda^\beta X^\alpha VU, \quad UX = \lambda^\gamma XU, \quad VW = \lambda^\delta WV \quad \text{and} \quad UW = \lambda^\epsilon WU,$$

Furthermore, for a suitable $\mathbb{C}$-valued cocycle on $\mathrm{H}_3(\alpha) \times \mathbb{Z}$,

$$\mathrm{A}_\theta^{5,3}(\alpha, \beta, \gamma, \delta, \epsilon) \cong C^*(\mathbb{C}, \mathrm{H}_3(\alpha) \times \mathbb{Z}).$$

(b) Let π' be a representation of $\mathrm{H}'_{5,3} = \mathrm{H}_{5,3}(\alpha, \beta, \gamma, \delta, \epsilon)$ such that $\pi = \pi'$ (as scalars) on the center $(\mathbb{Z}, 0, 0, 0, 0)$ of $\mathrm{H}'_{5,3}$, and let A be the C^*-algebra generated by π'. Then $\mathrm{A} \cong \mathrm{A}_\theta^{5,3}(\alpha, \beta, \gamma, \delta, \epsilon) = \mathrm{A}_\theta'^{5,3}$ (say) via a unique isomorphism ω such that the following diagram commutes.

$$\begin{array}{ccc} \mathrm{H}'_{5,3} & \xrightarrow{\pi} & \mathrm{A}_\theta'^{5,3} \\ \pi' \searrow & & \swarrow \omega \\ & \mathrm{A} & \end{array}$$

PROOF. To use Packer's result, we regard $\mathrm{H}_{5,3}(\alpha, \beta, \gamma, \delta, \epsilon)$ as an extension

$$\mathrm{H}_{5,3}(\alpha, \beta, \gamma, \delta, \epsilon) \cong \mathbb{Z} \times (0, \mathbb{Z}, \mathbb{Z}, \mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z} \times (\mathrm{H}_3(\alpha) \times \mathbb{Z})$$

(with $\mathrm{H}_3(\alpha) \cong (0, \mathbb{Z}, 0, \mathbb{Z}, \mathbb{Z}) \subset \mathrm{H}_{5,3}(\alpha, \beta, \gamma, \delta, \epsilon)$); this extension has cocycle

$$[s, s'] = [(j, k, m, n), (j', k', m', n')] = \lambda^{\gamma nj' + \alpha\gamma m'n(n-1)/2 + \beta nm' + \delta mk' + \epsilon nk'},$$

$$(\mathrm{H}_3(\alpha)\times\mathbb{Z}, \mathrm{H}_3(\alpha)\times\mathbb{Z}) \to \mathbb{T}.$$

The application of Packer's result requires the consideration of the related function

$$\chi^{s'}(s) = [s', s]\overline{[s, s^{-1}s's]}, \quad s, s' \in (0, \mathbb{Z}, \mathbb{Z}, \mathbb{Z}, \mathbb{Z}) \cong \mathrm{H}_3(\alpha)\times\mathbb{Z}.$$

It must be shown that $\chi^{s'}$ is non-trivial on the centralizer of s' in $\mathrm{H}_3(\alpha)\times\mathbb{Z}$ if s' has finite conjugacy class in $\mathrm{H}_3(\alpha)\times\mathbb{Z}$; this is easy because the only elements of $\mathrm{H}_3(\alpha)\times\mathbb{Z}$ that have finite conjugacy class are in the center $Z_1 = (\mathbb{Z}, \mathbb{Z}, 0, 0)$ of $\mathrm{H}_3(\alpha)\times\mathbb{Z}$, so their centralizer is all of $\mathrm{H}_3(\alpha)\times\mathbb{Z}$. Thus the C^*-crossed product $C^*(\mathbb{C}, \mathrm{H}_3(\alpha)\times\mathbb{Z})$ is simple; it is isomorphic to $\mathrm{A}^{5,3}_\theta(\alpha, \beta, \gamma, \delta, \epsilon)$ because, with basis members

$$e_1 = (1, 0, 0, 0), \quad e_2 = (0, 1, 0, 0), \quad e_3 = (0, 0, 1, 0) \quad \text{and} \quad e_4 = (0, 0, 0, 1)$$

for $\mathrm{H}_3(\alpha)\times\mathbb{Z}$, the unitaries

$$U' = \delta_{e_4}, \quad V' = \delta_{e_3}, \quad W' = \delta_{e_2} \quad \text{and} \quad X' = \delta_{e_1}$$

in $\ell_1(\mathrm{H}_3(\alpha)\times\mathbb{Z}) \subset C^*(\mathbb{C}, \mathrm{H}_3(\alpha)\times\mathbb{Z})$ satisfy (CR). $\square$

The theorem showed $\mathrm{A}^{5,3}_\theta(\alpha, \beta, \gamma, \delta, \epsilon)$ was simple by showing it was isomorphic to the simple C^*-crossed product $C^*(\mathbb{C}, \mathrm{H}_3(\alpha)\times\mathbb{Z})$. It follows that $\mathrm{A}^{5,3}_\theta(\alpha, \beta, \gamma, \delta, \epsilon)$ is isomorphic to a number of other C^*-crossed products (much as in [3; Theorem 3]), one of which has been derived from a flow at the beginning of the section for the case $\epsilon = 0$. Here are 2 other C^*-crossed products that can be used to establish the simplicity of $\mathrm{A}^{5,3}_\theta(\alpha, \beta, \gamma, \delta, \epsilon)$. The variable change for the second one will be used again, in the proof of Theorem 3 below.

1. Take the C^*-algebra B generated by U, V and X from (CR), satisfying

$$UV = \lambda^\beta X^\alpha VU, \quad UX = \lambda^\gamma XU, \quad \text{and} \quad VX = XV.$$

The algbera B is a faithful simple quotient of a discrete cocompact subgroup $\mathrm{H}_4(\beta, \alpha, \gamma)$ of H_4, the connected 4-dimensional nilpotent group [6; Theorem 2]. Then the rest of (CR) gives an action of $\mathbb{Z}$ on B generated by Ad_W; it follows that $\mathrm{A}^{5,3}_\theta(\alpha, \beta, \gamma, \delta, \epsilon) \cong C^*(B, \mathbb{Z})$. The simplicity of $C^*(B, \mathbb{Z})$ can be proved directly by showing that $(\mathrm{Ad}_W)^r = \mathrm{Ad}_{W^r}$ is outer on B if $r \neq 0$ [8].

2. First, we change the variables in (CR). Pick relatively prime integers c, d such that $d\delta + c\epsilon = 0$ and let a, b be integers such that $ad - bc = 1$. Put

$$U' = U^aV^b \quad \text{and} \quad V' = U^cV^d.$$

Then keeping X and W the same, (CR) becomes

$$(\mathrm{CR}') \qquad \begin{cases} U'V' = \lambda^{\beta'} X^\alpha V'U', & U'X = \lambda^{a\gamma} XU', \\ U'W = \lambda^{\delta'} WU', & \text{and} \quad V'X = \lambda^{c\gamma} XV' \end{cases}$$

(other pairs of unitaries commuting) for some integer β' and $\delta' = b\delta + a\epsilon$. Note that $\mathrm{A}^{5,3}_\theta(\alpha, \beta, \gamma, \delta, \epsilon)$ is generated by U', V', W, X (since $ad - bc = 1$). Consider the C^*-algebra $B' = \mathrm{A}_{\theta'} \otimes \mathcal{C}(\mathbb{T})$ generated by unitaries V', W and X_1 satisfying

$$V'X = \lambda^{c\gamma} XV', \quad V'W = WV' \quad \text{and} \quad WX_1 = X_1W;$$

here $e^{2\pi i\theta'} = \lambda^{c\gamma}$. Then the rest of (CR$'$) gives an action of $\mathbb{Z}$ on B' generated by $\mathrm{Ad}_{U'}$, and it follows that $C^*(B', \mathbb{Z}) \cong \mathrm{A}^{5,3}_\theta(\alpha, \beta, \gamma, \delta, \epsilon)$. One can prove the simplicity of $C^*(B', \mathbb{Z})$ directly; the method of proof is to show that the Connes spectrum of $\mathrm{Ad}_{U'}$ is $\mathbb{T}$, which follows from Theorem 2 and [11; 8.11.12].

§4. Other Simple Quotients of $C^*(\mathrm{H}_{5,3}(\alpha,\beta,\gamma,\delta,\epsilon))$.

Now assume that λ is a primitive qth root of unity and that U, V, W and X are unitaries generating a simple quotient A of $C^*(\mathrm{H}_{5,3}(\alpha,\beta,\gamma,\delta,\epsilon))$, i.e., they satisfy

$$\text{(CR)} \qquad UV=\lambda^\beta X^\alpha VU, \quad UX=\lambda^\gamma XU, \quad VW=\lambda^\delta WV \quad \text{and} \quad UW=\lambda^\epsilon WU.$$

We may assume that A is irreducibly represented. Then, if

$$(c') \qquad \begin{cases} q_1 \text{ is the order of } \lambda^\gamma \text{ and} \\ q_2 \text{ is the lcm of the orders of } \lambda^\delta \text{ and } \lambda^\epsilon, \end{cases}$$

W^{q_2} and X^{q_1} are scalar multiples of the identity (by irreducibility). Since W can be multiplied by a scalar without changing (CR), we may assume $W^{q_2}=1$. However, $X^{q_1}=\mu'$, a multiple of the identity. Put $X=\mu X_1$ for $\mu^{q_1}=\mu'$, so that $X_1^{q_1}=1$, and substitute $X=\mu X_1$ in (CR) to get

$$(\mathrm{CR}_1) \qquad \begin{cases} UV=\lambda^\beta\mu^\alpha X_1^\alpha VU, \quad UX_1=\lambda^\gamma X_1U, \\ VW=\lambda^\delta WV, \quad UW=\lambda^\epsilon WU \quad \text{and} \quad W^{q_2}=1=X_1^{q_1}. \end{cases}$$

1. If μ is also a root of unity, then (CR_1) (along with irreducibility) shows that U and V, as well as W and X, are (multiples of) unipotent unitaries, so A is finite dimensional.

2. If μ is not a root of unity, the flow $\mathcal{F}=(\mathbb{Z}^2,\mathbb{T}^2)$ used above to get a concrete representation of $\mathrm{A}_\theta^{5,3}(\alpha,\beta,\gamma,\delta,\epsilon)$ can be modified to get a concrete representation of A on $L^2(\mathbb{Z}_{q_1}\times\mathbb{T})$ (where $\mathbb{Z}_{q_1}$ is the subgroup of $\mathbb{T}$ with q_1 elements). The proof of the simplicity of A comes next.

First consider the universal C^*-algebra $\mathfrak{A}$ generated by unitaries satisfying

$$(\mathrm{CR}_1) \qquad \begin{cases} UV=\lambda^\beta\mu^\alpha X_1^\alpha VU, \quad UX_1=\lambda^\gamma X_1U, \\ VW=\lambda^\delta WV, \quad UW=\lambda^\epsilon WU \quad \text{and} \quad W^{q_2}=1=X_1^{q_1}. \end{cases}$$

A change of variables is useful. Pick relatively prime integers c,d such that $d\delta+c\epsilon=0$ and let a,b be integers such that $ad-bc=1$. Put

$$U'=U^aV^b \quad \text{and} \quad V'=U^cV^d.$$

Then keeping X and W the same, (CR_1) becomes

$$(\mathrm{CR}_2) \qquad \begin{cases} U'V'=\xi X_1^\alpha V'U', \quad U'X_1=\lambda^{a\gamma}X_1U', \\ U'W=\lambda^{\delta'}WU', \quad V'X=\lambda^{c\gamma}XV' \quad \text{and} \quad W^{q_2}=1=X_1^{q_1} \end{cases}$$

(other pairs of unitaries commuting), where $\xi=\lambda^\beta\mu^\alpha\lambda^s$ for some integer s, and $\delta'=b\delta+a\epsilon$. It is clear that $\lambda^{\delta'}$ is a primitive q_2-th root of unity and that the algebra $\mathfrak{A}$ is generated by U',V',W and X_1, since $ad-bc=1$.

Let $B = C^*(X_1, V')$ and let $C(\mathbb{Z}_{q_2}) = C^*(W)$ be the C^*-algebra generated by W. Since W commutes with X_1 and V', we can form the tensor product algebra $B \otimes C(\mathbb{Z}_{q_2}) = C^*(X_1, V', W)$. The automorphism $\mathrm{Ad}_{U'}$ acts on this tensor product as $\sigma \otimes \tau$, where σ and τ are automorphisms of B and $C(\mathbb{Z}_{q_2})$, respectively, given by

$$\sigma(X_1) = \lambda^{a\gamma} X_1, \quad \sigma(V') = \xi_1 X_1^{\alpha} V' \quad \text{and} \quad \tau(W) = \zeta W.$$

Therefore, by the universality of $\mathfrak{A}$ and of the C^*-crossed product $C^*(B \otimes C(\mathbb{Z}_{q_2}), \mathbb{Z})$, these algebras are isomorphic. By Rieffel's Proposition 1.2 [14], the latter of these is isomorphic to $M_{q_2}(D)$, where $D = C^*(B, \mathbb{Z}) = C^*(X_1, V', U'^{q_2})$, and the action of $\mathbb{Z}$ on B is generated by σ^{q_2}.

Now, the unitaries X_1, V' and U'^{q_2} generating D satisfy

$$(\star) \qquad \begin{cases} U'^{q_2} V' = \xi^{q_2} \lambda^{s'} X_1^{\alpha q_2} V' U'^{q_2}, \quad V' X_1 = \lambda^{c\gamma} X_1 V', \\ U'^{q_2} X_1 = \lambda^{a\gamma q_2} X_1 U'^{q_2} \quad \text{and} \quad X_1^{q_1} = 1, \end{cases}$$

for some $s' \in \mathbb{Z}$.

Now we apply another change of variables. Choose relatively prime integers c', d' such that $cd' + aq_2c' = 0$, then pick integers a', b' with $a'd' - b'c' = 1$, and put

$$U'' = U'^{q_2 a'} V'^{b'} \quad \text{and} \quad V'' = U'^{q_2 c'} V^{d'}.$$

Then $(\star)$ becomes (keeping X_1 the same)

$$(\star\star) \qquad \begin{cases} U''V'' = \xi_1 X_1^{\alpha q_2} V'' U'', \qquad V'' X_1 = X_1 V'', \\ U'' X_1 = \lambda' X_1 U'' \quad \text{and} \quad X_1^{q_1} = 1, \end{cases}$$

where $\xi_1 = \xi^{q_2} \lambda^{s'}$ for some integer s', $\lambda' = \lambda^{\gamma(aq_2a' + cb')}$ has order q_3 dividing q_1 (the order of λ^γ), and perhaps $q_3 \neq q_1$.

Now, with $\mathbb{Z}_{q_1} \subset \mathbb{T}$ representing the subgroup with q_1 members, one observes that D is isomorphic to the crossed product of $C^*(C(\mathbb{Z}_{q_1} \times \mathbb{T}), \mathbb{Z})$ from the flow generated by $\phi(w, v) = (\lambda' w, \xi_1 \lambda^{\gamma \alpha q_2} v)$. (Note that the flow is not minimal unless the order of λ' is exactly q_1.) This proves the following.

3. Theorem. *The universal C^*-algebra $\mathfrak{A}$ generated by unitaries U, V, W and X_1 satisfying* (CR_1) *as for* 2 *(see also* (c')*) is isomorphic to $M_{q_2}(D)$, where $D = C^*(C(\mathbb{Z}_{q_1} \times \mathbb{T}), \mathbb{Z})$, as above.*

Therefore, we now obtain all simple algebras satisfying (CR_1).

4. Corollary. *Every simple C^*-algebra generated by unitaries satisfying* (CR_1) *is isomorphic to a matrix algebra over an irrational rotation algebra.*

PROOF. By Theorem 3, any such simple algebra Q is a quotient of $M_{q_2}(D)$. Hence $Q = M_{q_2}(Q')$ where Q' is a simple quotient of D. But such a Q' is generated by unitaries satisfying $(\star\star)$, but with X_1 (of order q_1) replaced by another unitary X_2, which after suitable rescaling, has order equal to the order of λ'. But this algebra is known to be a matrix algebra over an irrational rotation algebra (see for example Theorem 3 of [4]). $\square$

We state

5. Theorem. *A C^*-algebra* A *is isomorphic to a simple infinite dimensional quotient of* $C^*(\mathrm{H}_{5,3}(\alpha,\beta,\gamma,\delta,\epsilon))$ *if and only if* A *is isomorphic to* $\mathrm{A}_\theta^{5,3}(\alpha,\beta,\gamma,\delta,\epsilon)$ *for an irrational* θ*, or to an algebra as in Corollary* 4.

The matrix algebra presentation for the simple C^*-algebra A generated by unitaries satisfying (CR_1) is not given in a definite form in the proof of Corollary 4. Here is an explicit matrix presentation. First, as near the beginning of the section, change the variables so that (CR_1) becomes

$$(\mathrm{CR}_2) \qquad \begin{cases} U'V' = \xi X_1^\alpha V'U', \quad U'X_1 = \lambda^{a\gamma} X_1 U', \\ U'W = \lambda^{\delta'} WU', \quad V'X = \lambda^{c\gamma} XV' \quad \text{and} \quad W^{q_2} = 1 = X_1^{q_1} \end{cases}$$

Now we shall present the algebra A by unitaries in a matrix algebra as follows. Consider the C^*-algebra B_1 generated by unitaries u, v and x enjoying the relations

$$uv = \xi' x^{q_2\alpha} vu, \;\; ux = \lambda^{q_2 a\gamma} xu, \;\; vx = \lambda^{c\gamma} xv \text{ and } x^{q_3} = 1,$$

where q_3 is the least common multiple of the orders of $\lambda^{q_2 a\gamma}$ and $\lambda^{c\gamma}$, and ξ' is to be determined below. Clearly, q_3 divides q_1 so that also $x^{q_1} = 1$. It was shown in the proof of Theorem 6.4 of [6] that B_1 is isomorphic to a $q_3 \times q_3$ matrix algebra over an irrational rotation algebra, when ξ' is not a root of unity. Hence it will suffice to show that A is isomorphic to $M_{q_2}(B)$ (so that $Q = q_2 q_3$). Indeed, let

$$U' = \begin{pmatrix} 0 & 0 & \cdots & 0 & u \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

(so $U'\mathrm{diag}(K_1, K_2, \ldots, K_{q_2})U'^* = \mathrm{diag}(uK_{q_2}u^*, K_1, K_2, \ldots, K_{q_2-1})$),

$$\begin{aligned} V' &= \mathrm{diag}(\tau_1 v,\; \tau_2 x^{-\alpha} v,\; \tau_3 x^{-2\alpha} v, \ldots,\; \tau_{q_2} x^{-(q_2-1)\alpha} v), \\ W &= \mathrm{diag}(1, \zeta^{-1}, \zeta^{-2}, \ldots, \zeta^{-(q_2-1)}), \\ X &= \mathrm{diag}(x,\; \lambda^{-a\gamma} x,\; \lambda^{-2a\gamma} x, \ldots,\; \lambda^{-(q_2-1)a\gamma} x), \end{aligned}$$

where

$$\tau_j = \lambda^{a\alpha\gamma j(j-1)/2}\, \xi_1^{-j+1}, \;\; 1 \le j \le q_2, \text{ and } \tau_{q_2}\xi'\lambda^{-q_2(q_2-1)\alpha^2\gamma} = \xi_1.$$

One now checks that these unitaries satisfy (CR_2). It is also evident that they generate $M_{q_2}(B)$. $\square$

§**5.** ***K*-Theory and the Trace Invariant.** In this section we shall calculate the K-groups of the C^*-algebra $A := \mathrm{A}_\theta^{5,3}(\alpha,\beta,\gamma,\delta,\epsilon)$ by means of the Pimsner-Voiculescu six term exact sequence [13]. Since one of the groups in the sequence turns out to have torsion elements, the application of this result requires careful examination.

6. Theorem. *For the C^*-algebra $\mathrm{A}_\theta^{5,3}(\alpha,\beta,\gamma,\delta,\epsilon)$, one has $K_0 = K_1 = \mathbb{Z}^6 \oplus \mathbb{Z}_\alpha$.*

PROOF. To prove this theorem, we combine two applications of the PV sequence corresponding to two presentations P1 and P2 of A as follows.

P1. In view of (CR), let $B_1 = C^*(X, V, U)$ and let Ad_W, with

$$\mathrm{Ad}_W(X) = X, \qquad \mathrm{Ad}_W(V) = \lambda^{-\delta} V, \qquad \mathrm{Ad}_W(U) = \lambda^{-\epsilon} U,$$

generate an action of $\mathbb{Z}$ on B_1, so that $A = C^*(B_1, \mathbb{Z})$. Applying the PV sequence to B_1, viewed as the crossed product of $C(\mathbb{T}^2) = C^*(X, V)$ by the automorphism Ad_U, it is not hard to see that $K_0(B_1) = \mathbb{Z}^3$ and $K_1(B_1) = \mathbb{Z}^3 \oplus \mathbb{Z}_\alpha$. Since Ad_W is homotopic to the identity, the PV sequence immediately gives

$$K_1(A) = \mathbb{Z}^6 \oplus \mathbb{Z}_\alpha.$$

However, since in the short exact sequence

$$0 \longrightarrow K_0(B_1) \xrightarrow{i_*} K_0(A) \xrightarrow{\delta} K_1(B_1) \longrightarrow 0$$

$K_1(B_1)$ has torsion, we cannot readily obtain $K_0(A)$. For this, the next presentation will help.

P2. In view of (CR), we can also let $B_2 = C^*(X, V, W) = C(\mathbb{T}) \otimes A_{\delta\theta}$, where $C(\mathbb{T}) = C^*(X)$ and $A_{\delta\theta} = C^*(V, W)$. Let $\sigma = \mathrm{Ad}_U$, with

$$\sigma(X) = \lambda^\gamma X, \qquad \sigma(V) = \lambda^\beta X^\alpha V, \qquad \sigma(W) = \lambda^\epsilon W,$$

generate an action of $\mathbb{Z}$ on B_2, so that $A = C^*(B_2, \mathbb{Z})$. In this case the PV sequence becomes

$$(*) \qquad \begin{array}{ccccc} K_0(B_2) & \xrightarrow{id_* - \sigma_*} & K_0(B_2) & \xrightarrow{i_*} & K_0(A) \\ {\scriptstyle\delta_1}\big\uparrow & & & & \big\downarrow{\scriptstyle\delta_0} \\ K_1(A) & \xleftarrow{i_*} & K_1(B_2) & \xleftarrow{id_* - \sigma_*} & K_1(B_2) \end{array}$$

It is not hard to see that a basis for $K_1(B_2) = \mathbb{Z}^4$ is given by $\{[X], [V], [W], [\xi]\}$ where $\xi = X \otimes e + 1 \otimes (1 - e)$ and $e = e(V, W)$ is a Rieffel projection in $A_{\delta\theta}$ of trace $\delta\theta \mod 1$. Also, a basis of $K_0(B_2) = \mathbb{Z}^4$ is given by $\{[1], [e], B_{XV}, B_{XW}\}$ where $B_{XV} = [P_{XV}] - [1]$ is the Bott element in X, V and P_{XV} the usual Bott projection in the commuting variables X, V. The action of $id_* - \sigma_*$ on $K_1(B_2)$ is given by

$$id_* - \sigma_* : \qquad [X] \mapsto 0, \qquad [V] \mapsto -\alpha[X], \qquad [W] \mapsto 0, \qquad [\xi] \mapsto m\alpha[X]$$

for some integer m, as shown by the following lemma. The action of $id_* - \sigma_*$ on $K_0(B_2)$ is given by

$$id_* - \sigma_* : \qquad [1] \mapsto 0, \qquad [e] \mapsto \alpha B_{XW}, \qquad B_{XW} \mapsto 0, \qquad B_{XV} \mapsto 0.$$

The action on $[e]$ is also shown in the following

7. Lemma. *We have $\sigma_*[e] = [e] - \alpha B_{XW}$ in $K_0(B_2)$, and $\sigma_*[\xi] = [\xi] + m\alpha[X]$ for some integer m.*

PROOF. The proof of the first equality can be established using an argument quite similar to that of the proof of Lemma 4.2 of [15]. Hence the kernel of $id_* - \sigma_*$ on $K_0(B_2)$ is $\mathbb{Z}^3$. For the second equality, let $\eta = (id_* - \sigma_*)[\xi]$. From P1 and (*) we have

$$\mathbb{Z}^6 \oplus \mathbb{Z}_\alpha = K_1(A) = \mathbb{Z}^3 \oplus Im(i_*) = \mathbb{Z}^3 \oplus \frac{K_1(B_2)}{Im(id_* - \sigma_*)} = \mathbb{Z}^3 \oplus \frac{K_1(B_2)}{\mathbb{Z}\alpha[X] + \mathbb{Z}\eta}.$$

Thus

$$\frac{K_1(B_2)}{\mathbb{Z}\alpha[X] + \mathbb{Z}\eta} = \mathbb{Z}^3 \oplus \mathbb{Z}_\alpha. \tag{**}$$

But since $K_1(B_2) = \mathbb{Z}^4$, it follows that the subgroup $\mathbb{Z}\alpha[X] + \mathbb{Z}\eta$ must have rank one.[1] Therefore, $\mathbb{Z}\alpha[X] + \mathbb{Z}\eta = \mathbb{Z}d[X]$ for some integer d. Substituting this into (**) one gets $d = \alpha$ and so $\eta \in \mathbb{Z}\alpha[X]$. $\square$

It now follows that in $K_1(B_2)$ one has $Im(id_* - \sigma_*) = \mathbb{Z}\alpha[X]$ and that $Ker(id_* - \sigma_*) = \mathbb{Z}^3$ whether m is zero or not. Therefore, from the exactness of (*) we obtain $Im(\delta_0) = \mathbb{Z}^3$ and hence by Lemma 7

$$K_0(A) = \mathbb{Z}^3 \oplus Im(i_*) = \mathbb{Z}^3 \oplus \frac{K_0(B_2)}{Im(id_* - \sigma_*)} = \mathbb{Z}^3 \oplus \frac{K_0(B_2)}{\mathbb{Z}\alpha B_{XW}} = \mathbb{Z}^6 \oplus \mathbb{Z}_\alpha$$

which completes the proof of Theorem 6. $\square$

The Trace Invariant.

8. Theorem. *The range of the unique trace on $K_0(\mathrm{A}_\theta^{5,3}(\alpha,\beta,\gamma,\delta,\epsilon))$ is $\mathbb{Z} + \mathbb{Z}\rho\theta + \mathbb{Z}\gamma\delta\theta^2$ where $\rho = \gcd\{\gamma,\delta,\epsilon\}$.*

Note that this agrees with the the trace invariant $\mathbb{Z} + \mathbb{Z}\theta + \mathbb{Z}\theta^2$ of the algebra $\mathrm{A}_\theta^{5,3}$ as done in [15], section 2, in the case $(\alpha,\beta,\gamma,\delta,\epsilon) = (1,0,1,1,0)$.

PROOF. First we make an appropriate change of variables for the unitary generators of the algebra $A = \mathrm{A}_\theta^{5,3}(\alpha,\beta,\gamma,\delta,\epsilon)$. Referring back to the defining relations (CR), pick integers a, b, c, d such that $b\delta + a\epsilon = 0$, $ad - bc = 1$, and let

$$U' = U^a V^b, \qquad V' = U^c V^d.$$

Then the commutation relations (CR), with W remaining the same and X suitably scaled, become

$$\begin{aligned} &U'V' = X^\alpha V'U', \quad U'X = \lambda^{a\gamma} XU', \quad V'W = \lambda^{d\delta + c\epsilon} WV', \\ &U'W = WU', \qquad V'X = \lambda^{c\gamma} XV', \quad WX = XW \end{aligned}$$

[1] If $0 \to F_1 \to G \to F_2 \oplus H \to 0$ is a short exact sequence of finitely generated Abelian groups, where F_1, F_2 are free groups and H is torsion, then $rank(G) = rank(F_1) + rank(F_2)$. This can be seen from the naturally obtained short exact sequence $0 \to F_1 \oplus F_2 \to G \to H \to 0$, from which the result follows. (If G has rank greater than that of a subgroup K, then G/K contains a non-torsion element.)

Let $B = C^*(X, U', V')$. It is isomorphic to the crossed product of $C^*(X, U') = A_{a\gamma\theta}$ by $\mathbb{Z}$ and automorphism $\mathrm{Ad}_{V'}$. An easy application of Pimsner's trace formula shows that

$$\tau_* K_0(B) = \mathbb{Z} + \mathbb{Z}a\gamma\theta + \mathbb{Z}c\gamma\theta = \mathbb{Z} + \mathbb{Z}\gamma\theta,$$

since $(a, c) = 1$. Next, it is not hard to see that an application of the Pimsner-Voiculescu sequence to the above crossed product presentation of B gives the basis $\{[X], [V'], [U'], [\xi]\}$ for $K_1(B)$, where $[X]$ has order α, $\xi = 1 - e + ew^*V'^*e$ is a unitary in B, e is a Rieffel projection in $A_{a\gamma\theta}$ of trace $(a\gamma\theta) \mathrm{mod}\, 1$, and w is a unitary in $A_{a\gamma\theta}$ such that $V'^*eV' = wew^*$ (which exists by Rieffel's Cancellation Theorem [14]). The underlying connecting homomorphism $\partial : K_1(B) \to K_0(A_{a\gamma\theta})$ gives $\partial[\xi] = [e]$ and $\partial[V'] = [1]$, the usual basis of $K_0(A_{a\gamma\theta})$.

To apply Pimsner's trace formula, one calculates the usual "determinant" on the aforementioned basis, since the kernel of $id_* - (\mathrm{Ad}_W)_*$ is all of $K_1(B)$ (since Ad_W is homotopic to the identity). It is easy to see that this determinant (whose values are in $\mathbb{R}/\tau_*K_0(B)$) on the elements $[X], [V'], [U']$ gives the respective values $1, (d\delta + c\epsilon)\theta, 1$. For the ξ, since now Ad_W fixes $A_{a\gamma\theta}$ (and in particular e and w), one obtains

$$\mathrm{Ad}_W(\xi)\xi^* = (1 - e + \lambda^{d\delta+c\epsilon}ew^*V'^*e)(1 - e + eV'we) = 1 - e + \lambda^{d\delta+c\epsilon}e.$$

Now a simple homotopy path connecting this element to 1 is just $t \mapsto 1 - e + e^{2\pi i\theta(d\delta+c\epsilon)t}e$, and the corresponding determinant gives the value $(d\delta + c\epsilon)\theta\tau(e)$. Since $\tau(e) = a\gamma\theta \mod 1$, the range of the trace is

$$\tau_*K_0(A) = \mathbb{Z} + \mathbb{Z}\gamma\theta + \mathbb{Z}(d\delta + c\epsilon)\theta + \mathbb{Z}\gamma a(d\delta + c\epsilon)\theta^2.$$

Now $a(d\delta + c\epsilon) = ad\delta + ac\epsilon - c(b\delta + a\epsilon) = \delta$, and similarly $-b(d\delta + c\epsilon) = \epsilon$, thus showing that $d\delta + c\epsilon = \gcd\{\delta, \epsilon\}$. Therefore, one gets $\tau_*K_0(A) = \mathbb{Z} + \mathbb{Z}\gcd\{\gamma, \delta, \epsilon\}\theta + \mathbb{Z}\gamma\delta\theta^2$. $\square$

Discussion of Classification.

Next, let us consider briefly the classification of the algebras $\mathrm{A}_\theta^{5,3}(\alpha, \beta, \gamma, \delta, \epsilon)$. First, it is easy to show that $\mathrm{A}_\theta^{5,3}(\alpha, \beta, \gamma, \delta, \epsilon) \cong \mathrm{A}_{-\theta}^{5,3}(\alpha, \beta, \gamma, \delta, \epsilon)$. Second, we note that the simple quotients $\mathrm{A}_\theta^{5,3} = \mathrm{A}_\theta^{5,3}(1, 0, 1, 1, 0)$ have been almost completely classified in [15]; specifically, they have been classified for all non-quartic irrationals (which are those that are not zeros of any polynomial of degree at most 4 with integer coefficients). But generally, with $\lambda = e^{2\pi i\theta}$ for an irrational θ, the operator equations

$$\text{(CR)} \qquad UV = \lambda^\beta X^\alpha VU, \quad UX = \lambda^\gamma XU, \quad VW = \lambda^\delta WV \quad \text{and} \quad UW = \lambda^\epsilon WU,$$

for $\mathrm{A}_\theta^{5,3}(\alpha, \beta, \gamma, \delta, \epsilon)$ can be modified by changing some of the variables, i.e., by substituting $X_0 = e^{2\pi i\theta\beta/\alpha}X$ and putting $\lambda_0 = \lambda^\rho$, where $\rho = \gcd\{\gamma, \delta, \epsilon\}$, and then $\gamma_0 = \gamma/\rho$, $\delta_0 = \delta/\rho$ and $\epsilon_0 = \epsilon/\rho$ with $\gcd\{\gamma_0, \delta_0, \epsilon_0\} = 1$. The equations (CR) become

$$(\mathrm{CR}_0) \qquad \begin{cases} UV = X_0^\alpha VU, \quad UX_0 = \lambda_0^{\gamma_0} X_0U, \quad VW = \lambda_0^{\delta_0} WV \quad \text{and} \\ UW = \lambda_0^{\epsilon_0} WU \quad \text{with} \quad \gcd\{\gamma_0, \delta_0, \epsilon_0\} = 1, \end{cases}$$

which are the equations for $\mathrm{A}^{5,3}_{\rho\theta}(\alpha,0,\gamma_0,\delta_0,\epsilon_0)$, so

$$\mathrm{A}^{5,3}_{\theta}(\alpha,\beta,\gamma,\delta,\epsilon) \cong \mathrm{A}^{5,3}_{\rho\theta}(\alpha,0,\gamma_0,\delta_0,\epsilon_0)$$

where $\gcd\{\gamma_0,\delta_0,\epsilon_0\} = 1$. This reduces the classification to the class of algebras $\mathrm{A}^{5,3}_{\theta}(\alpha,0,\gamma,\delta,\epsilon)$ where $\gcd\{\gamma,\delta,\epsilon\} = 1$.

If two such C^*-algebras $A_j = \mathrm{A}^{5,3}_{\theta_j}(\alpha_j,0,\gamma_j,\delta_j,\epsilon_j)$, $j = 1,2$, are isomorphic, where now $\rho_j = \gcd\{\gamma_j,\delta_j,\epsilon_j\} = 1$, what constraints must hold between their respective parameters? As we observed in Theorem 6, one must have $\alpha_1 = \alpha_2$. By Theorem 8, one has

$$\mathbb{Z} + \mathbb{Z}\theta_1 + \mathbb{Z}\gamma_1\delta_1\theta_1^2 = \mathbb{Z} + \mathbb{Z}\theta_2 + \mathbb{Z}\gamma_2\delta_2\theta_2^2.$$

One can show that if one assumes that θ_j are non-quadratic irrationals, then these trace invariants are equal if, and only if, there is a matrix $S \in \mathrm{GL}(2,\mathbb{Z})$ such that

$$\begin{pmatrix} \theta_2 \\ \gamma_2\delta_2\theta_2^2 \end{pmatrix} = S \begin{pmatrix} \theta_1 \\ \gamma_1\delta_1\theta_1^2 \end{pmatrix} \bmod \begin{pmatrix} \mathbb{Z} \\ \mathbb{Z} \end{pmatrix}.$$

Further, one can more easily show that if θ_j are non-quartic irrationals (i.e., not roots of polynomials over $\mathbb{Z}$ of degree at most four), then the trace invariants are equal if, and only if,

$$\theta_2 = (\pm\theta_1) \bmod 1, \qquad \text{and} \qquad \gamma_2\delta_2\theta_2^2 = (\pm\gamma_1\delta_1\theta_1^2 + m\theta_1) \bmod 1,$$

for some integer m. An interesting special situation can be considered. For example, if one fixes θ (assumed non-quartic for simplicity) and varies the other parameters, then the above shows that $\gamma_1\delta_1 = \gamma_2\delta_2$ will follow from $A_1 \cong A_2$. At this point it is not clear if the parameters can be determined more precisely than this. For example, is it possible, if θ is held fixed, that the equalities $\gamma_1 = \gamma_2$ and $\delta_1 = \delta_2$ could fail to hold? This is unclear. However, the following heuristic argument (based only on canonical considerations) suggests that perhaps γ_j, δ_j are uniquely determined.

Let us attempt to apply a canonical transformation of the unitary generators of the form

$$U_1 = U^{a_1}V^{b_1}W^{c_1}, \qquad V_1 = U^{a_2}V^{b_2}W^{c_2}, \qquad W_1 = U^{a_3}V^{b_3}W^{c_3},$$

in the hope of changing γ,δ, by working out the commutation relations and ensuring that they are preserved. (We have kept X the same since it is the only auxiliary unitary that can occur if one looks at the most general transformation of the form $U_1 = U^{a_1}V^{b_1}W^{c_1}X^{d_1}$, $V_1 = U^{a_2}V^{b_2}W^{c_2}X^{d_2}$ — in fact, the commutator $[U_1,V_1]$ is a scalar multiple of $X^{\alpha(a_1b_2-a_2b_1)}$.) The 3 by 3 matrix T with rows a_j, b_j, c_j should have determinant ±1 for the new unitaries to generate the same C^*-algebra. The first relation in (CR) demands that $a_1b_2 - a_2b_1 = 1$ so as to keep X^α the same. Also, since $VX = XV$ and $WX = XW$ must be preserved, we should have $V_1X = XV_1$ and $W_1X = XW_1$. However, it is easy to see that these imply that $a_2 = 0$ and $a_3 = 0$, respectively (since U does not commute with X). Since the relation between V_1 and W_1 does not contain X, one must have $a_2b_3 - a_3b_2 = 0$, which is already satisfied. Similarly, for the relation between U_1 and W_1 one must have $a_1b_3 - a_3b_1 = 0$. But

since $a_3 = 0$ this gives $b_3 = 0$. This means that the matrix T is upper triangular with 1 or -1 on its diagonal, hence the transformation is reduced to

$$U_1 = U^{\pm 1} V^{b_1} W^{c_1}, \qquad V_1 = V^{\pm 1} W^{c_2}, \qquad W_1 = W^{\pm 1}.$$

(where the third $\pm$ here is independent of the first two, which should both be 1 or both -1). In view of this transformation, however, the new commutation relations are now forced to take the following form

$$U_1 V_1 = X^{\alpha} V_1 U_1, \quad U_1 X = \lambda^{\pm\gamma} X U_1, \quad V_1 W_1 = \lambda^{\pm\delta} W_1 V_1, \quad U_1 W_1 = \lambda^{\delta b_1 \pm \epsilon} W_1 U_1,$$

(and of course $V_1 X = X V_1$, $W_1 X = X W_1$, and after one rescales X). These are exactly in the same form as the relations (CR). In particular, the integer parameters γ and δ, since they are assumed to be positive, have remained unchanged. (Also unchanged is ϵ, since it is, by $(*)$ of Theorem 1, smaller than δ.) This seems to suggest that γ and δ (and hence also ϵ) are uniquely determined in an isomorphism classification theorem. The broad scope of this classification problem, however, must be left to another time; the fact that γ and δ are not clearly singled out in the invariants considered here, but appear mixed, seems to present an obstacle to the classification of these C^*-algebras. (The authors doubt that the Ext invariant of Brown-Douglas-Fillmore contains any more information, though they have not checked this in detail.)

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